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DYNAMIC BUCKLING OF A CYLINDRICAL SHELL WITH VARIABLE THICKNESS SUBJECT TO A TIME-DEPENDENT EXTERNAL PRESSURE VARYING AS A POWER FUNCTION OF TIME

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In this study, the dynamic buckling of an elastic cylindrical shell with variable thickness, subject to a uniform external pressure which is a power function of time, has been considered. Initially, the fundamental relations and Donnell-type dynamic buckling equation of an elastic cylindrical shell with variable thickness have been obtained. Then, employing Galerkin's method, these equations have been reduced to a time-dependent differential equation with variable coefficients. Finally, applying a special Ritz-type method, the critical static and dynamic loads, the corresponding wave numbers and dynamic factor have been found analytically. Using those results, the effects of the variation of the thickness with a linear, a parabolic or an exponential function in the axial direction and the effect of the variation of the power of time in the external pressure expression are studied using pertinent computations. It is observed that these effects change appreciably the critical parameters of the problem. The present method has been verified, comparing the results of the present work and those of previous works in the literature, for a shell with constant thickness subject to a uniform external pressure varying linearly with time.

1. INTRODUCTION

Thin shells are used to a great extent as basic structural parts in simple and complex structural systems, particularly in the aerospace industry. There are numerous methods to determine the values of static critical loads, consistent with experimental results, for shells with constant thickness under different loads with different boundary conditions. One such method, widely used due to its high efficiency, has been given by Sachenkov and Baktieva [1]. On the other hand, there are fewer research studies on shells with variable thickness, due to the difficulties in production and theoretical analysis. Nonetheless, it is highly probable that this type of structural parts will be used more in the future due to the advantages of their low weights and small dimensions and the progress in fabrication

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methods. In recent years, numerous research works have been published, concerning the buckling vibration of shells with variable thickness. In those studies, generally, Kirchhoff–Love's first approximation theory has been used and the effect of the variation of thickness on the critical parameters has been proved numerically [2–8].

The effect of the variation of thickness on the dynamic buckling load has not been studied appreciably. The solution of a dynamic problem is reduced to the determination of the dynamic factor for certain loading cases. The dynamic factor can be found, using different methods, depending on the manner in which the loading is applied, particularly on the loading speed. In this respect, the method given by Sachenkov and Baktieva [1] for the solution of the dynamic stability problem under impulsive time-dependent loads plays an important role in the verification problems of the buckling of plates and shells.

The aim of the present study is to investigate the dynamic buckling of an elastic cylindrical shell with continuously varying thickness, subject to a uniform external pressure which is a power function of time, using the method given by Sachenkov and Baktieva [1].

2. FUNDAMENTAL RELATIONS AND GOVERNING EQUATIONS

Consider an elastic circular cylindrical shell, with immovable hinged supports at the ends, length L, radius R and thickness h, varying continuously in the axial direction. The right-handed system of Cartesian co-ordinates is selected in such a way that the origin is on the middle surface, the z-axis is perpendicular to the middle surface of the plate, positive inwards, and the x- and y-axis are in the axial and tangential directions, respectively (Figure 1).

According to Kirchhoff–Love's first approximation theory, the strain at distance z from the middle surface is given as follows:

$$[\varepsilon_x, \varepsilon_y, \varepsilon_{xy}] = \left[\varepsilon_x^0 - z\frac{\partial^2 w}{\partial x^2}, \varepsilon_y^0 - z\frac{\partial^2 w}{\partial y^2}, \varepsilon_{xy}^0 - z\frac{\partial^2 w}{\partial x \partial y}\right], \tag{1}$$

where w is the displacement of the middle surface of the shell in the normal direction, positive inwards, ε_x^0 and ε_y^0 are the normal strains in the directions of the Ox- and Oy-axis, respectively, and ε_{xy}^0 is the shear strain, all on the same surface.

The stress–strain relations for the cylindrical shell are given as follows:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix}, \tag{2}$$

where E is the elasticity modulus and ν is the Poisson ratio of the material. The internal forces and moments per unit length of the cross-section of the shell can be expressed as follows:

$$[(T_x, T_y, T_{xy}), (M_x, M_y, M_{xy})] = \int_{-h/2}^{h/2} [1, z][\sigma_x, \sigma_y, \sigma_{xy}]\, \mathrm{d}z. \tag{3}$$

The internal forces (T_x, T_y, T_{xy}) are given in terms of the stress function $\bar{\Phi} = \Phi/h_0$ as follows:

$$[T_x, T_y, T_{xy}] = \left[\frac{\partial^2 \Phi}{\partial y^2}, \frac{\partial^2 \Phi}{\partial x^2}, -\frac{\partial^2 \Phi}{\partial x \partial y}\right], \tag{4}$$

where h_0 is the nominal thickness of the shell.

Substituting equations (1–4) into the dynamic stability and compatibility equations of the cylindrical shell [8, 9], the following equations are obtained for a cylindrical shell of variable thickness subject to a uniform external pressure which varies as a power function of time:

$$\nabla^2\nabla^2 w + \frac{2}{h^3}\frac{\mathrm{d}h^3}{\mathrm{d}x}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) + \frac{1}{h^3}\frac{\mathrm{d}^2 h^3}{\mathrm{d}x^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$-\frac{1}{R}\frac{12(1-\nu^2)}{Eh^3}\frac{\partial^2 \Phi}{\partial x^2} + R(P_1 + P_0 t^\alpha)\frac{12(1-\nu^2)}{Eh^3}\frac{\partial^2 w}{\partial y^2} + \frac{12(1-\nu^2)\rho}{Eh^2}\frac{\partial^2 w}{\partial t^2} = 0, \quad (5)$$

$$\nabla^2\nabla^2\Phi + 2h\frac{\mathrm{d}h^{-1}}{\mathrm{d}x}\left(\frac{\partial^3 \Phi}{\partial x^3} + \frac{\partial^3 \Phi}{\partial x \partial y^2}\right) + h\frac{\mathrm{d}^2 h^{-1}}{\mathrm{d}x^2}\left(\frac{\partial^2 \Phi}{\partial x^2} - \nu\frac{\partial^2 \Phi}{\partial y^2}\right) + \frac{Eh}{R}\frac{\partial^2 w}{\partial x^2} = 0, \quad (6)$$

where the bi-harmonic Laplace operator is defined as $\nabla^2\nabla^2 = \partial^4/\partial x^4 + 2\,\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$ and P_1, P_0, ρ, t and α are, respectively, the static external pressure, the loading speed, the density of the material, the time and the power of time in the external pressure expression, the last one being equal to or greater than unity.

3. THE SOLUTION OF THE DIFFERENTIAL EQUATIONS

Assuming the shell to be simply supported along the peripheries of both bases, the solution of the system of equations (5–6) is sought as follows [1]:

$$\Phi = \phi(t)\sin m\pi\bar{x}\sin n\bar{y}, \qquad w = \xi(t)\sin m\pi\bar{x}\sin n\bar{y}, \quad \bar{x} = x/L,\ \bar{y} = y/R, \quad (7)$$

where m is the half wave number in the axial direction, n is the wave number in the circumferential direction and $\phi(t)$ and $\xi(t)$ are time-dependent amplitudes. Substituting expressions (7) into equation set (5–6), applying Galerkin's method in the ranges $0 \leqslant \bar{x} \leqslant 1$ and $0 \leqslant \bar{y} \leqslant 2\pi$, eliminating $\phi(t)$ from the equations, thus obtained, and recalling that when the half wave number m is equal to one, the wave number n for a shell of medium length satisfies the inequality $n^4 \gg (\pi R/L)^4$, the following equation is obtained [1]:

$$\frac{\mathrm{d}^2\xi(\tau)}{\mathrm{d}\tau^2} + \frac{V^2}{C_3 R^4}\left[\frac{C_1}{12(1-\nu^2)}n^4 + \frac{\pi^4 R^6}{4C_2 L^4 n^4} - \frac{(P_1 + P_0 t_{kr}^\alpha \tau^\alpha)R^3}{2E}n^2\right]t_{kr}^2\xi(\tau) = 0, \quad (8)$$

where V is the speed of sound in isotropic material, satisfying the relation $\rho = E/V^2$, $t = \tau t_{kr}$, in which t_{kr} is the critical time and the dimensionless time parameter τ satisfies $0 \leqslant \tau \leqslant 1$, and the following definitions apply:

$$C_1 = \int_0^1 h^3(\bar{x})\sin^2\pi\bar{x}\,\mathrm{d}\bar{x}, \quad C_2 = \int_0^1 h^{-1}(\bar{x})\sin^2\pi\bar{x}\,\mathrm{d}\bar{x}, \quad C_3 = \int_0^1 h(\bar{x})\sin^2\pi\bar{x}\,\mathrm{d}\bar{x}. \quad (9)$$

Applying the method in reference [1] (see Appendix A) to differential equation (8), the following characteristic equation is obtained:

$$P_0 t_{kr}^\alpha = B_0(\alpha)\left[\frac{EC_1}{6(1-\nu^2)R^3}n^2 + \frac{E\pi^4 R^3}{2C_2 L^4}\frac{1}{n^6} - P_1\right] + \frac{2B_1(\alpha)EC_3 R}{t_{kr}^2 V^2 n^2}, \quad (10)$$

where $B_0(\alpha)$ and $B_1(\alpha)$ are defined as follows:

$$B_0(\alpha) = \frac{\int_0^1 [\xi(\tau)]^2 \,\mathrm{d}\tau}{2\int_0^1 \int_0^\tau \eta^\alpha \xi'(\eta)\xi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau}, \quad B_1(\alpha) = \frac{\int_0^1 [\xi'(\tau)]^2 \mathrm{d}\tau}{2\int_0^1 \int_0^\tau \eta^\alpha \xi'(\eta)\xi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau}. \tag{11}$$

The approximating function, satisfying the initial conditions $\xi(0) = 0$, $\xi'_\tau(1) = 0$, has been chosen as

$$\xi(\tau) = A\mathrm{e}^{50\tau}\tau[52/51 - \tau] \tag{12}$$

in the first approximation [1]. For the minimization of $P_0 t_{kr}^\alpha$ with respect to n^2, the following equation is obtained for determining the critical load:

$$P_0 t_{kr}^\alpha = B_0(\alpha)\left[\frac{EC_1}{3(1-v^2)R^3}\,n^2 - \frac{E\pi^4 R^3}{C_2 L^4}\,\frac{1}{n^6} - P_1\right]. \tag{13}$$

Considering $P_0 \geqslant 200$ MPa/s^α and $P_1 = 0$ and eliminating t_{kr} from equations (10) and (13), the wave number corresponding to the dynamic critical load is expressed by the following equation [10]:

$$n_d^2 = \frac{3^{1/4}(1-v^2)^{1/4}R^{3/2}\pi}{L(C_1C_2)^{1/4}}\,\Lambda^{\alpha/(2+2\alpha)} \tag{14}$$

where the following definition applies:

$$\Lambda = \frac{4B_1(\alpha)C_3P_0^{2/\alpha}R^{(3+\alpha)/\alpha}L^{2(1+\alpha)/\alpha}C_2^{(1+\alpha)/(2\alpha)}[3(1-v^2)]^{(3+\alpha)/(2\alpha)}}{[B_0(\alpha)]^{(2+\alpha)/\alpha}\pi^{2(1+\alpha)/\alpha}E^{2/\alpha}C_1^{(3+\alpha)/(2\alpha)}V^2}. \tag{15}$$

Substituting $P_1 = 0$ and equation (14) into equation (13), the dynamic critical load is

$$P_{kr}^d = P_0 t_{kr}^\alpha = \frac{B_0(\alpha)C_1^{3/4}E\pi}{[3(1-v^2)]^{3/4}C_2^{1/4}R^{3/2}L}\,\Lambda^{\alpha/(2+2\alpha)}. \tag{16}$$

In the static case ($t_{kr} \to \infty$, $P_0 \to 0$), the wave number corresponding to this case is given by the following equation:

$$n_{st}^2 = \frac{3^{1/2}\pi L R^{3/2}(1-v^2)^{1/4}}{(C_1C_2)^{1/4}}. \tag{17}$$

Substituting equation (17) into equation (13) and replacing $P_{kr}^d/B_0(\alpha)$ by P_{kr}^{st}, the static critical load is as follows:

$$P_{kr}^{st} = \frac{2\pi E C_1^{3/4}}{3^{3/2}(1-v^2)^{3/4}C_2^{1/4}R^{3/2}L}. \tag{18}$$

Thus, the dynamic factor can be found from its definition $K_d = P_{kr}^d / P_{kr}^{st}$ as

$$K_d = \frac{LC_2^{1/2}}{\pi} \left[\frac{3^{(3+2\alpha)/\alpha}}{2^{2/\alpha}} \frac{B_0(\alpha) B_1(\alpha) C_3 P_0^{2/\alpha} R^{(3+\alpha)/\alpha} (1-\nu^2)^{(3+\alpha)/\alpha}}{E^{2/\alpha} C_1^{(3+\alpha)/(2\alpha)} V^2} \right]^{\alpha/(2+2\alpha)}. \tag{19}$$

For the special case, when the external pressure varies linearly, i.e., $\alpha = 1$, and the thickness is constant, equations (16), (18), (19) yield the respective expressions as follows [1]:

$$P_{kr}^d = 3{\cdot}696 \frac{h}{R} \left[\frac{RP_0}{V} \frac{E}{(1-\nu^2)^{1/2}} \right]^{1/2}, \tag{20}$$

$$P_{kr}^{st} = 0{\cdot}886 \frac{E}{(1-\nu^2)^{3/4}} \frac{R}{L} \left(\frac{h}{R} \right)^{5/2}, \tag{21}$$

$$K_d = 4{\cdot}208 \left(\frac{1-\nu^2}{E} \frac{P_0 R}{V} \frac{RL^2}{h^3} \right)^{1/2}. \tag{22}$$

4. TWO DIFFERENT CASES OF THICKNESS VARIATION

In this section, some special cases of thickness variation, considered in previous literature, will be mentioned, i.e., linear, parabolic and exponential variations (Figure 1).

4.1. CYLINDRICAL SHELL WITH THICKNESS VARYING AS A POWER FUNCTION IN THE AXIAL DIRECTION

The case of a cylindrical shell in which the thickness changes as a power function of time has been considered by Irie *et al.* [5]. The thickness at the two ends being h_1 and h_2, this case can be expressed in the following manner:

$$h = h_2 - (h_2 - h_1)(1 - \bar{x})^{\beta}, \quad \beta > 0. \tag{23}$$

For extremely large and small values of β, the thickness changes abruptly near the ends causing a special difficulty in obtaining precise values in numerical computations. Nonetheless, this kind of shell is very rare in applications. As shown in Figure 1, $\beta = 1, 2$ correspond to the cases of linear and parabolic variations respectively.

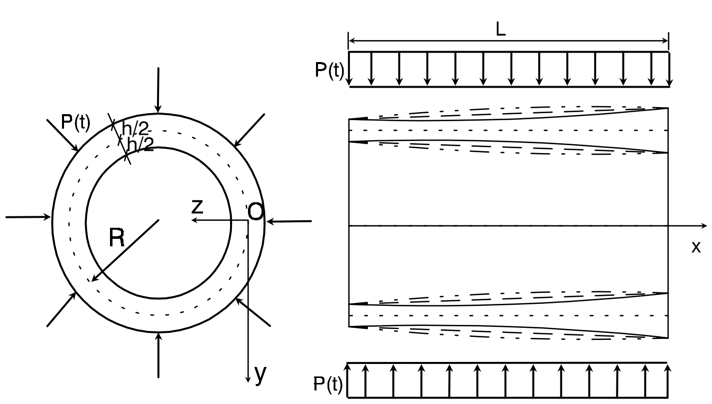

Figure 1. Elastic cylindrical shell of variable thickness h subject to a uniform time-dependent external pressure: – – –, h linear; —··—, h parabolic; ——, h exponential.

4.2. CYLINDRICAL SHELL WITH THICKNESS VARYING AS AN EXPONENTIAL FUNCTION IN THE AXIAL DIRECTION

As shown in Figure 1, the case of cylindrical shells with thickness varying as an exponential function has been considered (see also reference [5]). In this case the thickness is expressed in the following manner:

$$h = h_2(h_1/h_2)^{(1-\bar{x})}. \tag{24}$$

5. NUMERICAL COMPUTATIONS AND RESULTS

Numerical computations have been carried out for the following data used in experimental tests [11] and theoretical computations [1, 9, 12]:

Material properties, $E = 7{\cdot}75 \times 10^4$ MPa, $\nu = 0{\cdot}3$.
Shell properties, $h_2 = 8 \times 10^{-4}$ m, $R = 9 \times 10^{-2}$ m, $L = 0.2$ m.
Velocity of sound in the material, $V = 5 \times 10^3$ m/s.

Using above data, numerical results found for various loading speeds and α values have been presented in graphical and tabular forms in the sequel.

Figure 2 shows the variations of the dynamic critical load and dynamic factor with the ratio h_1/h_2. In shells with thickness varying as a linear, parabolic or exponential function in the axial direction, an increase in the ratio h_1/h_2 causes an increase in the dynamic critical load, whilst there is a decrease in the dynamic factor. When the foregoing ratio becomes unity, the latter quantities assume the values for a shell of constant thickness. The highest effects on the dynamic critical load and dynamic factor take place for an exponential function.

Figure 3 shows the variations of the dynamic critical load and dynamic factor with the ratio R/h_2, when the shell parameters are constant or varying in the axial direction. In this example, the shell parameters are taken as $R = 0{\cdot}1$ m and $L = 0{\cdot}3$ m. As the ratio R/h_2 increases, so does the dynamic critical load, whereas the dynamic factor decreases. Furthermore, in all the cases considered, including the case of constant thickness, the critical factors undergo relevant variations.

Table 1 shows the variations in the dynamic critical load and the corresponding values of the wave number and dynamic factor with the ratio h_1/h_2, for different loading speeds, in shells having thickness changing as a linear, parabolic or exponential function in the axial direction. When the foregoing ratio becomes unity, the values for a shell of constant

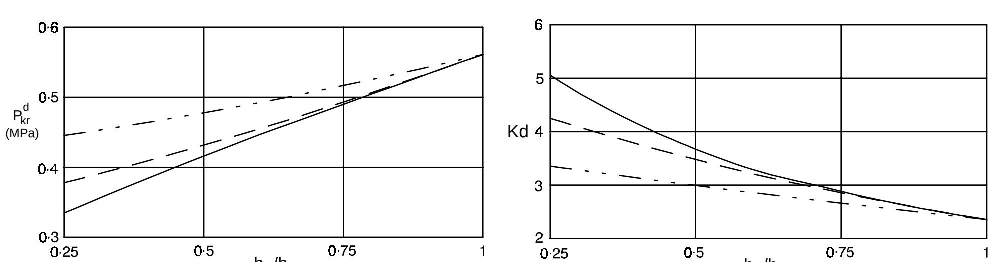

Figure 2. Variations of dynamic critical load and dynamic factor with the ratio h_1/h_2: – – –, h linear; —··—, h parabolic; ——, h exponential.

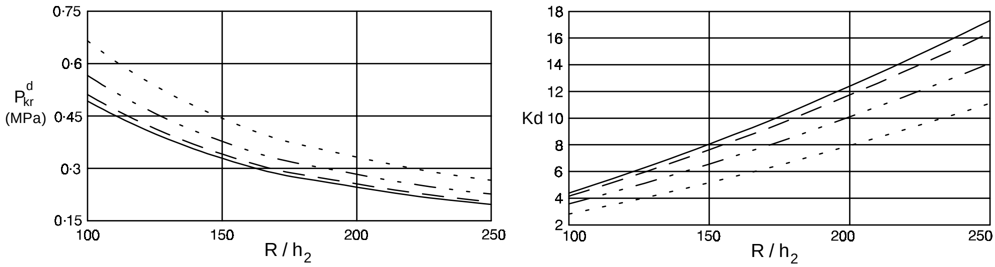

Figure 3. Variations of the dynamic critical load and dynamic factor with the ratio R/h_2: – – –, h linear; —··—, h parabolic; ——, h exponential; ·······, h constant, P_0 = 200 MPa/s, h_1/h_2 = 0·5, α = 1, R = 0·1 m, L = 0·3 m.

TABLE 1

Variations of the critical parameters with h_1/h_2 ($\alpha = 1$)

	P_0 = 200 MPa/s			P_0 = 470 MPa/s			P_0 = 650 MPa/s		
h_1/h_2	P_{kr}^d	K_d	n_d	P_{kr}^d	K_d	n_d	P_{kr}^d	K_d	n_d
$h = h_2(h_1/h_2)^{(1-\bar{x})}$									
0·25	0·3347	5·0590	11	0·5131	7·7553	14	0·6034	9·1202	15
0·50	0·4160	3·6756	9	0·6377	5·6346	12	0·7499	6·6263	13
0·75	0·4899	2·8860	8	0·7510	4·4241	10	0·8831	5·2028	11
1·00	0·5611	2·3557	7	0·8601	3·6124	9	1·0115	4·2482	9
$h = h_2 - (h_2/h_1)(1-\bar{x})^{\beta}$, $\beta = 1$									
0·25	0·3780	4·2507	10	0·5795	6·5162	13	0·6815	7·6630	14
0·50	0·4316	3·4806	9	0·6617	5·3357	11	0·7781	6·2748	12
0·75	0·4932	2·8567	8	0·7561	4·3793	10	0·8892	5·1501	11
1·00	0·5611	2·3565	7	0·8601	3·6124	9	1·0115	4·2482	9
$h = h_2 - (h_2/h_1)(1-\bar{x})^{\beta}$, $\beta = 2$									
0·25	0·4446	3·3577	9	0·6815	5·1473	11	0·8014	6·0532	12
0·50	0·4778	2·9944	8	0·7324	4·5903	10	0·8613	5·3982	11
0·75	0·5167	2·6648	7	0·7921	4·0851	9	0·9315	4·8041	10
1·00	0·5611	2·3565	7	0·8601	3·6124	9	1·0115	4·2482	9

thickness are obtained, as expected. It is observed that the critical parameters increase with an increase in the loading speed.

Table 2 shows the values of the dynamic critical load and the corresponding wave number and dynamic factor, for a shell with thickness varying as a linear, parabolic or exponential function pertaining to $h_1/h_2 = 0{\cdot}5$, for different values of the power of time, α, in the external pressure expression. In the case of variable thickness, the critical values decrease with increasing α and are appreciably different when compared to the case of constant thickness.

To verify the present method, Table 3 shows the results found in the present work for the case of constant thickness, along with the corresponding theoretical values given by Sachenkov and Baktieva [1], Wolmir [9] and Tazyukov [12] and the experimental ones given by Mineev [11]. Despite the differences in the methods of approach, the results of the present method are in quite good agreement with those of previous authors for a shell with constant thickness ($h_1/h_2 = 1$). Comparisons could not be made with those of Ershov *et al.* [2] since they did not give any numerical values, with Tonin and Bies [3] and Bergman

TABLE 2

Variations of the critical parameters with α ($P_0 = 650$ MPa/s$^{\alpha}$)

	$\alpha = 1$			$\alpha = 2$			$\alpha = 3$		
h_1/h_2	P^d_{kr}	K_d	n_d	P^d_{kr}	K_d	n_d	P^d_{kr}	K_d	n_d
$h = h_2(h_1/h_2)^{(1-\bar{x})}$									
0·50	0·7499	6·6263	13	0·0805	0·7116	4	0·0269	0·2373	6
1·00	1·0115	4·2482	9	0·1200	0·5041	5	0·0421	0·1767	1
$h = h_2 - (h_2/h_1)(1-\bar{x})^{\beta}$, $\beta = 1$									
0·50	0·7781	6·2748	12	0·0846	0·6822	4	0·0284	0·2289	2
1·00	1·0115	4·2482	9	0·1200	0·5041	3	0·0421	0·1767	1
$h = h_2 - (h_2/h_1)(1-\bar{x})^{\beta}$, $\beta = 2$									
0·50	0·8613	5·3982	11	0·0969	0·6071	3	0·0331	0·2072	2
1·00	1·0115	4·2482	9	0·1200	0·5041	3	0·0421	0·1767	1

TABLE 3

Comparison of the critical parameters with previous theoretical and experimental results ($\alpha = 1$)

P_0 (MPa/s)	Reference [1] Theoretical		Reference [9] Theoretical		Reference [11] Experimental		Reference [12] Theoretical		Const. thickness, $h_1/h_2 = 1$	
	n_d	n_{st}	n_d	n_{st}	n_d	n_{st}	n_d	n_{st}	n_d	n_{st}
200	8	6	8	6	7	6	8	6	7	6
470	9	6	9	6	8	6	9	6	9	6
650	10	6	10	6	9	6	10	6	10	6
	P^d_{kr} (MPa)	K_d	P^d_{kr} (MPa)	K_d	P^d_{kr} (MPa)	K_d	P^d_{kr} (MPa)	K_d	P^d_{kr} (MPa)	K_d
200	0·590	2·560	0·520	2·180	0·670	2·600	0·550	—	0·561	2·357
470	0·880	3·240	0·740	3·100	0·890	3·240	0·860	—	0·860	3·612
650	1·040	4·100	0·880	3·600	1·060	4·000	1·010	—	1·012	4·248

et al. [4] since they worked on free vibration and with Koiter *et al.* [8] since they studied buckling under axial compression.

6. CONCLUSION

The dynamic buckling of a shell with continuously varying thickness has been studied employing the method of Sachenkov and Baktieva [1]. Initially, the fundamental relations and modified Donnell-type dynamic buckling equations have been derived for a shell with varying thickness subject to an external pressure which is a power function of time. Then, applying Galerkin's method, a time-dependent differential equation with constant coefficients has been obtained. Finally, applying the method given by Sachenkov and Baktieva [1], analytical expressions have been found for the static and dynamic critical loads and the corresponding wave numbers and dynamic factor. Using these results, the dependence of the critical parameters on the variation of the thickness with a linear, parabolic or exponential function in the axial direction, are studied numerically. Furthermore, the present method has been verified by comparisons of critical parameters with the theoretical and experimental data given in the literature for the case of a shell with constant thickness subject to a uniform external pressure which is a linear function of time.

The analytical formulae obtained in the present study can be used to find precise values for the critical parameters pertaining to elastic cylindrical shells with continuously varying thickness.

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APPENDIX A: METHOD OF SACHENKOV AND BAKTIEVA [1]

Reference [1] considers the dynamic buckling problem of cylindrical shells, subject to external pressure, axial pressure and torsional loading, which are linear functions of time, and the solution of this problem has been transformed to the solution of the second order linear differential equation with variable coefficients given below:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} + \lambda(\tau)\, t_{cr}^2\, \xi = 0, \quad t = \tau t_{cr},\ 0 \leqslant \tau \leqslant 1, \tag{A.1}$$

where λ is a linear function of time, depending on the geometrical and mechanical properties of the shell and the wave parameters. When $\tau = 1$, the (ξ, τ) curve has an open minimum and, hence, the ξ function satisfies the following initial conditions:

$$\xi(0) = \frac{\mathrm{d}\xi(1)}{\mathrm{d}\tau} = 0. \tag{A.2}$$

In reference [1], equation (A.1) is solved by a mixed variational method of Ritz method type. The approximation function is selected in the following form:

$$\xi = A\mathrm{e}^{\alpha_1 \tau} \tau [(\alpha_1 + 2)(\alpha_1 + 1)^{-1} - \tau]. \tag{A.3}$$

Under these circumstances, displacement–time curve (ξ–τ) possesses two different characteristic regions. In the first region, the inertia force acts opposite to the external load, whereas in the second region, it changes sign at some point and enhances the external load from that point on. Consequently, displacement amplitude ξ tends to infinity. When this problem is solved by Bubnov's method, the foregoing factors affect the results appreciably. When equation (A.1) is integrated with respect to τ in the interval $0 \leqslant \tau \leqslant 1$, the work of inertia forces adds up to zero. This causes the dynamic critical load to take an illogical value, which also renders a false appearance for the kinetic energy of the deformed system when Bubnov's method is applied on equation (A.1). In reference [1], to prevent the vanishing of the total work done by the inertia forces, equation (A.1) is first multiplied by $\mathrm{d}\xi/\mathrm{d}\tau$ and then integrated with respect to τ first from 0 to 1 and then from 0 to τ.

Generally, $\xi(\tau)$ is chosen in the form given by equation (A.3) and in the curve of the dynamic critical load versus α_1, there is a minimum at $\alpha_1 = 50$ in the range $0 < \alpha_1 < 55$.

APPENDIX B: NOMENCLATURE

E	elasticity modulus of the isotropic material
h	thickness of the shell
h_0	nominal thickness of the shell
h_1 and h_2	thicknesses of the shell at the two ends
K_d	dynamic factor
L	length of the shell
M_x, M_y, M_{xy}	internal moments per unit length of the cross-section of the shell
m	half-wave number in the axial direction
n	wave number in the circumferential direction
n_{st}	wave number corresponding to the static critical load
n_d	wave number corresponding to the dynamic critical load
Ox, Oy, Oz	Cartesian co-ordinate axes with the origin on the middle surface of the shell
P_{kr}^{st}	static critical load
P_{kr}^{d}	dynamic critical load
P_0	loading speed
P_1	static external pressure
R	average radius of the shell
T_x, T_y, T_{xy}	internal forces per unit length of the cross-section of the shell
t	time
t_{kr}	critical time
V	speed of sound in isotropic material
$\bar{x} = x/L$	dimensionless x co-ordinate
$\bar{y} = y/R$	dimensionless y co-ordinate
w	displacement of the middle surface in the inwards normal direction
α	power of time in the external pressure expression
$\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0$	strain components on the middle surface of the shell
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	strain components
v	the Poisson ratio of the isotropic material
τ	dimensionless time parameter
ρ	density of the material
$\sigma_x, \sigma_y, \sigma_{xy}$	stress components
$\bar{\Phi}$	stress function
$\xi(t)$, $\phi(t)$	time-dependent amplitudes